COMMON FIXED POINT THEOREM FOR TWO MAPPINGS IN bi-b-METRIC SPACE

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ABSTRACT. In this paper we have used the concept of bi-metric space and introduced the concept of bi-b-metric space. Our objective is to obtain the common fixed point theorems for two mappings on two different b-metric spaces induced on same set X. In this paper we prove that on the set X two b-metrics are defined to form two different b-metric spaces and the two mappings defined on X have unique common fixed point.

1. INTRODUCTION.

The Banach Contraction Mapping Principle is very useful theorem. Hence it is very popular tool in solving existence problems in many branches of mathematical analysis. Banach fixed point theorem has many applications inside and outside Mathematics. In 1989, an interesting concept of generalized b–metric spaces was introduced by Bakhtin [2]. In 1993 Czerwik [7] extended the results of b-metric spaces. Many researchers generalized the Banach fixed point theorem in b-metric space. Czerwik [8] 1998 presented the generalization of Banach fixed point theorem in b-metric spaces. The existence and uniqueness theorems in b-Metric Space

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was presented by Agrawal [1]. In 1968 Maia generalized the result of well known Banach Contraction Principle by taking two metrics on a set \(X\). Mishra [13] generalized the Maia’s fixed point theorem in bi-metric spaces. Soni [12] gave the fixed point theorem for mappings in bi-metric spaces.

We want to extend some fixed point theorems in bimetric spaces which are also valid in bi-\(b\)-metric spaces. Chopade [6] obtained common fixed point theorems for contractive type mappings in metric space. Roshan [10] gave the common fixed point of four maps in \(b\)-Metric space. Suzuki [11] obtained some basic inequalities on a \(b\)-Metric space and its applications.

2. Some Basic Definitions and Preliminaries

**Definition 2.1.** Let \(X\) be a non-empty set. A function \(\delta : X \times X \rightarrow \mathbb{R}\) is called as a metric provided that for all \(u, v, w \in X\),

\[
\begin{align*}
\text{i.} & \quad \delta(u, v) \geq 0, \\
\text{ii.} & \quad \delta(u, v) = 0 \text{ if and only if } u = v, \\
\text{iii.} & \quad \delta(u, v) = \delta(v, u), \\
\text{iv.} & \quad \delta(u, v) = \delta(u, w) + \delta(w, v).
\end{align*}
\]

A pair \((X, \delta)\) is called a metric space.

**Definition 2.2.** Let \(X\) be a non-empty set and \(s \geq 1\) be a given real number. A function \(\delta : X \times X \rightarrow \mathbb{R}\) is called as a \(b\)-metric provided that for all \(u, v, w \in X\),

\[
\begin{align*}
\text{i.} & \quad \delta(u, v) \geq 0, \\
\text{ii.} & \quad \delta(u, v) = 0 \text{ if and only if } u = v, \\
\text{iii.} & \quad \delta(u, v) = \delta(v, u), \\
\text{iv.} & \quad \delta(u, v) \leq s\{\delta(u, w) + \delta(w, v)\}.
\end{align*}
\]

A pair \((X, \delta)\) is called a \(b\)-metric space. It is clear that definition of \(b\)-metric space is an extension of usual metric space.

**Remark 2.1.** If \(s = 1\), then the \(b\)-metric space is a usual metric space.

**Example 1.** Let \((X, d)\) be a metric space and \(\delta(u, v) = (d(u, v))^p\), where \(p > 1\) is a real number. Clearly, \(\delta(u, v)\) is \(b\)-metric with \(s = 2^{p-1}\).
Example 2. Example 2.2: If \( X = \mathbb{R} \), be the set of real numbers and \( d(u, v) = |u - v| \) a usual metric, then \( \delta(u, v) = (u - v)^2 \) is a \( b \)-metric on \( \mathbb{R} \) with \( s = 2 \), but not a metric on \( \mathbb{R} \).

Example 3. By Boriceanu [2], Let \( M = \{0, 1, 2\} \) and \( \delta : M \times M \to \mathbb{R} \) is defined as,
\[
\delta(0, 2) = \delta(2, 0) = m \geq 2, \\
\delta(0, 1) = \delta(1, 0) = \delta(1, 2) = \delta(2, 1) = 1, \\
\delta(0, 0) = \delta(1, 1) = \delta(2, 2) = 0.
\]
Here, \( \delta(u, v) \) is a \( b \)-metric on \( M \) with \( s = \frac{m}{2} \).

Definition 2.3. Let \((X, \delta)\) be a \( b \)-metric space then a sequence \( \{u_n\} \) in \( X \) is called as convergent sequence if there exists \( u \in X \) such that for all \( \epsilon > 0 \) there exist \( n(\epsilon) \in \mathbb{N} \) such that \( n \geq n(\epsilon) \) we have, \( \delta(u_n, u) < \epsilon \). In this case we write \( \lim_{n \to \infty} u_n = u \).

Definition 2.4. Let \((X, \delta)\) be a \( b \)-metric space then a sequence \( \{u_n\} \) in \( X \) is called as Cauchy sequence if for all \( \epsilon > 0 \) there exist \( n(\epsilon) \in \mathbb{N} \) such that \( m, n \geq n(\epsilon) \) we have, \( \delta(u_n, u_m) < \epsilon \).

Definition 2.5. Let \((X, \delta)\) be a \( b \)-metric space then \( X \) is said to be complete if every Cauchy sequence in \( X \) is convergent sequence in \( X \).

3. Main Result

We use the following Lemma to prove the main result.

Lemma 3.1. [11] Let \((X, \delta)\) be a complete \( b \)-metric space and let \( \{x_n\} \) be a sequence in \( X \). Assume that there exist \( r \in [0, 1) \) satisfying
\[
\delta(x_{n+1}, x_{n+2}) \leq r\delta(x_n, x_{n+1}) \text{ for any } n \in \mathbb{N}.
\]
Then \( \{x_n\} \) is Cauchy sequence in \( X \).

Theorem 3.1. Let \((X, \delta_1, s)\) and \((X, \delta_2, t)\) be a bi-\( b \)-metric space, where, \( s \geq 1 \) and \( t \geq 1 \) such that,
\[
i. \ \delta_1(u, v) \leq \delta_2(u, v) \text{ for all } u, v \in X.
\]
ii. $S : X \to X$ and $T : X \to X$ be any two selfmaps on $X$ satisfying,
\[
\delta_2(Su, Tv) \leq \alpha \frac{\delta_2(u, Su) \delta_2(v, Tv)}{\delta_2(v, Tv) + \delta_2(v, Su)} \\
+ \beta \frac{\delta_2(u, v)[1 + \delta_2(u, Su) + \delta_2(v, Su)]}{1 + \delta_2(u, v) + \delta_2(u, Su) \delta_2(v, Su) \delta_2(v, Tv)} \\
+ \gamma \frac{[\delta_2(u, Su) \delta_2(u, Tv)]}{\delta_2(u, v)},
\]
where, $\alpha, \beta, \gamma \in [0, 1]$ are such that $\alpha + \beta + 2\gamma \tau < 1$

iii. There exist a point $u_0 \in X$ such that the sequence $\{u_n\}$ of iterates defined as $u_1 = Su_0, u_2 = Tu_1, \ldots, u_{2n} = Tu_{2n-1}, u_{2n+1} = Su_{2n}$ for any $n \in N$ has a convergent subsequence $u_{n_k}$ converging to $u^*$ in $(X, \delta_1)$.

iv. Both the mappings $T$ and $S$ are continuous in $(X, \delta_1)$. Then, $T$ and $S$ have unique common fixed point in $X$.

**Proof.**

**Existence:** Given, $u_0 \in X$, and $\{u_n\}$ be a sequence of iterates in $X$ defined as
\[
S(u_{2n}) = u_{2n+1} \quad \text{and} \quad T(u_{2n-1}) = u_{2n}, \; n = 1, 2, \ldots.
\]

Using equation (3.1.1) and (3.1.2) we obtain that,
\[
\begin{align*}
\delta_2(u_{2n+1}, u_{2n+2}) & = \delta_2(Su_{2n}, Tu_{2n+1}) \\
& \leq \alpha \frac{\delta_2(u_{2n}, Su_{2n}) \delta_2(u_{2n+1}, Tu_{2n+1})}{\delta_2(u_{2n+1}, Tu_{2n+1}) + \delta_2(u_{2n+1}, Su_{2n})} \\
& + \beta \frac{\delta_2(u_{2n}, u_{2n+1})[1 + \delta_2(u_{2n}, Su_{2n}) + u_{2n+1}, Su_{2n}]}{1 + \delta_2(u_{2n}, u_{2n+1}) + \delta_2(u_{2n}, Su_{2n}) \delta_2(u_{2n+1}, Su_{2n}) \delta_2(u_{2n+1}, Tu_{2n+1})} \\
& + \gamma \frac{[\delta_2(u_{2n}, Su_{2n}) \delta_2(u_{2n}, Tu_{2n+1})]}{\delta_2(u_{2n}, u_{2n+1})} \\
& \leq \alpha \frac{\delta_2(u_{2n}, u_{2n+1}) \delta_2(u_{2n+1}, u_{2n+2})}{\delta_2(u_{2n+1}, u_{2n+2})} \\
& + \beta \frac{\delta_2(u_{2n}, u_{2n+1})[1 + \delta_2(u_{2n}, u_{2n+1})]}{1 + \delta_2(u_{2n}, u_{2n+1})} \\
& + \gamma \frac{[\delta_2(u_{2n}, u_{2n+1}) \delta_2(u_{2n}, u_{2n+2})]}{\delta_2(u_{2n}, u_{2n+1})} \\
& \leq \alpha \delta_2(u_{2n}, u_{2n+1}) + \beta \delta_2(u_{2n}, u_{2n+1}) + \gamma \delta_2(u_{2n}, u_{2n+2}) \\
& \leq (\alpha + \beta) \delta_2(u_{2n}, u_{2n+1}) + \gamma \delta_2(u_{2n}, u_{2n+1}) + \gamma \delta_2(u_{2n+1}, u_{2n+2})
\end{align*}
\]
\[ (1 - \gamma t)\delta_2(u_{2n+1}, u_{2n+2}) \leq (\alpha + \beta + \gamma t)\delta_2(u_{2n}, u_{2n+1}) \]
\[ \delta_2(u_{2n+1}, u_{2n+2}) \leq \frac{(\alpha + \beta + \gamma t)}{(1 - \gamma t)}\delta_2(u_{2n}, u_{2n+1}) \]
\[ \delta_2(u_{2n+1}, u_{2n+2}) \leq r\delta_2(u_{2n}, u_{2n+1}), \]

where \( r = \frac{(\alpha + \beta + \gamma t)}{(1 - \gamma t)} < 1 \). In general, for all \( n \in \mathbb{N} \),
\[ (3.1.3) \quad \delta(u_{n+1}, u_{n+2}) \leq r\delta(u_n, u_{n+1}), \]

where \( r = \frac{(\alpha + \beta + \gamma t)}{(1 - \gamma t)} < 1 \).

Therefore by Lemma 3.1 the sequence \( \{u_n\} \) is Cauchy Sequence in \( X \). Since the cauchy sequence \( \{u_n\} \) defined by (3.1.2) has convergent subsequence \( \{u_{nk}\} \) in \( (X, \delta_1) \) converging to \( u^* \) in \( (X, \delta_1) \), the sequence \( \{u_n\} \) also converges to \( u^* \) in \( (X, \delta_1) \). Hence,
\[ \lim_{n \to \infty} u_n = \lim_{n \to \infty} u_{2n} = \lim_{n \to \infty} u_{2n-1} = \lim_{n \to \infty} u_{2n+1} = u^*. \]

Now we show that \( u^* \) is fixed point of both the mappings \( S \) and \( T \). As \( S \) and \( T \) are continuous in \( (X, \delta_1) \), therefore,
\[ S(u^*) = S[\lim_{n \to \infty} u_{2n}] = \lim_{n \to \infty} [Su_{2n}] = u^*. \]
Similarly,
\[ T(u^*) = T[\lim_{n \to \infty} u_{2n-1}] = \lim_{n \to \infty} [Tu_{2n-1}] = u^*. \]
Thus \( u^* \) is common fixed point of the mappings \( S \) and \( T \).

**Uniqueness:**
Suppose, \( u^* \) and \( v^* \) be two common fixed points of the mappings \( T \) and \( S \). Therefore, \( Su^* = Tu^* = u^* \) and \( Sv^* = Tv^* = v^* \). Consider:
\[ \delta_2(u^*, v^*) = \delta_2(Su^*, Tv^*), \]
\[ \delta_2(u^*, v^*) \leq \frac{\alpha \delta_2(u^*, Su^*) \delta_2(v^*, Tv^*)}{\delta_2(v^*, Tv^*) + \delta_2(v^*, Su^*)} + \frac{\beta}{1 + \delta_2(v^*, v) + \delta_2(u^*, Su^*) \delta_2(v^*, Tu^*) \delta_2(v^*, Tv^*)} \]
\[ + \frac{\gamma \delta_2(u^*, Su^*) \delta_2(u^*, Tv^*)}{\delta_2(u^*, v^*)}. \]
\[ \delta_2(u^*, v^*) \leq \alpha \frac{\delta_2(u^*, u^*) \cdot \delta_2(v^*, v^*)}{\delta_2(v^*, v^*) + \delta_2(v^*, u^*)} + \beta \frac{\delta_2(u^*, v^*) [1 + \delta_2(u^*, u^*) + \delta_2(v^*, u^*)]}{1 + \delta_2(u^*, v^*) + \delta_2(u^*, u^*) \cdot \delta_2(v^*, u^*) \cdot \delta_2(v^*, v^*)} + \gamma \frac{\delta_2(u^*, v^*)}{\delta_2(u^*, v^*)} \]

But, \( \beta < 1 \). Therefore we have, \( \delta(u^*, v^*) < \delta(u^*, v^*) \). This is contradiction, Hence, \( T \) and \( S \) have unique common fixed point in \( X \).

This completes the proof. \( \square \)

**Corollary 3.1.** If Conditions (i), (ii) and (iv) of theorem (3.1) holds and moreover \( (X, \delta_1) \) is complete \( b \)-metric space then \( T : X \to X \) and \( S : X \to X \) have unique common fixed point in \( X \).

**Theorem 3.2.** Let \( (X, \delta_1, s) \) and \( (X, \delta_2, t) \) be a bi \( b \)-metric space, where, \( s \geq 1 \) and \( t \geq 1 \). Let \( P = \{ T_i : i \in I, \text{the set of positive integers} \} \) be a family of mappings on \( X \) such that the following conditions holds

i. \( \delta_1(u, v) \leq \delta_2(u, v) \) for all \( u, v \in X \).

ii. \( X \) is complete with respect to \( \delta_1 \).

iii. for each \( T_j : X \to X \in P \) there exist \( T_i : X \to X \in P \) such that

\[
\delta_2(T_i^m u, T_j^n v) \leq \alpha \frac{\delta_2(u, v) \cdot \delta_2(v, T_j^n v)}{\delta_2(u, T_i^m u) + \delta_2(v, T_i^m u)} + \beta \frac{\delta_2(u, v) [1 + \delta_2(u, T_i^m u) + \delta_2(v, T_i^m u)]}{1 + \delta_2(u, v)} + \gamma \frac{\delta_2(u, T_i^m u) \cdot \delta_2(v, T_j^n v)}{\delta_2(u, v)},
\]

\( (3.2.1) \)

where, \( m, n \) are positive integers and \( \alpha, \beta, \gamma \in [0, 1) \) are such that \( \alpha + \beta + 2\gamma t < 1 \).

iv. mapping \( T_i \) is continuous in \( (X, \delta_1) \) for all \( i \in I \), then \( P \) has a unique common fixed point.
Proof.

Existence: Given, \( u_0 \in X \). We define a sequence of iterates \( \{ u_n \} \) in \( X \) as

\[
(3.2.2) \quad u_{2n-1} = T_i^m(u_{2n-2}) \quad \text{and} \quad u_{2n} = T_j^n(u_{2n-1}), n = 1, 2, \ldots.
\]

Using equation (3.2.1) and (3.2.2) we obtain that,

\[
\begin{align*}
\delta_2(u_{2n+1}, u_{2n+2}) &= \delta_2(T_i^m u_{2n}, T_j^n u_{2n+1}) \\
&\leq \alpha \frac{\delta_2(u_{2n}, u_{2n+1}), \delta_2(u_{2n+1}, T_j^n u_{2n+1})}{\delta_2(u_{2n}, T_i^m u_{2n}) + \delta_2(u_{2n+1}, T_i^m u_{2n})} \\
&\quad + \beta \frac{\delta_2(u_{2n}, u_{2n+1})[1 + \delta_2(u_{2n}, T_i^m u_{2n}) + \delta_2(u_{2n+1}, T_i^m u_{2n})]}{1 + \delta_2(u_{2n}, u_{2n+1})} \\
&\quad + \gamma \frac{\delta_2(u_{2n}, u_{2n+1}), \delta_2(u_{2n}, u_{2n+2})}{\delta_2(u_{2n}, u_{2n+1})} \\
&\leq \alpha \delta_2(u_{2n+1}, u_{2n+2}) + \beta \frac{\delta_2(u_{2n}, u_{2n+1})[1 + \delta_2(u_{2n}, u_{2n+1})]}{1 + \delta_2(u_{2n}, u_{2n+1})} + \gamma \delta_2(u_{2n}, u_{2n+2}) \\
&\leq \alpha \delta_2(u_{2n+1}, u_{2n+2}) + \beta \delta_2(u_{2n}, u_{2n+1}) + \gamma \delta_2(u_{2n}, u_{2n+2}) \\
&\leq \alpha \delta_2(u_{2n+1}, u_{2n+2}) + \beta \delta_2(u_{2n}, u_{2n+1}) + \gamma t \delta_2(u_{2n}, u_{2n+1}) + \gamma \delta_2(u_{2n+1}, u_{2n+2}) \\
&\leq (1 - \alpha - \gamma t) \delta_2(u_{2n+1}, u_{2n+2}) \leq (\beta + \gamma t) \delta_2(u_{2n}, u_{2n+1}) \\
&\leq \frac{\beta + \gamma t}{1 - \alpha - \gamma t} \delta_2(u_{2n}, u_{2n+1}) \\
&\leq r \delta_2(u_{2n+1}, u_{2n+2}),
\end{align*}
\]

where \( r = \frac{\beta + \gamma t}{1 - \alpha - \gamma t} < 1 \). In general, for all \( n \in \mathbb{N} \),

\[
(3.2.3) \quad \delta(u_{n+1}, u_{n+2}) \leq r \delta(u_n, u_{n+1}),
\]

where \( r = \frac{\beta + \gamma t}{1 - \alpha - \gamma t} < 1 \). Therefore, by Lemma 3.1 the sequence \( \{ u_n \} \) is Cauchy Sequence in \( X \). Since, the sequence \( \{ u_n \} \) defined by (3.2.2) is a cauchy sequence in
Thus, \(\lim_{n \to \infty} u_n = \lim_{n \to \infty} u_{2n} = \lim_{n \to \infty} u_{2n-1} = \lim_{n \to \infty} u_{2n+1} = u^*\).

Now we show that \(u^*\) is a fixed point of both the mappings \(T_i^m\) and \(T_j^n\).

As \(T_i^m\) and \(T_j^n\) are continuous in \((X, \delta_1)\), therefore,

\[
T_i^m(u^*) = T_i^m[\lim_{n \to \infty} u_{2n}] = \lim_{n \to \infty} [T_i^m u_{2n}] = u^*.
\]

Similarly,

\[
T_j^n(u^*) = T_j^n[\lim_{n \to \infty} u_{2n-1}] = \lim_{n \to \infty} [T_j^n u_{2n-1}] = u^*.
\]

Thus, \(u^*\) is a common fixed point of the mappings \(T_i^m\) and \(T_j^n\).

**Uniqueness:**

Suppose, \(u^*\) and \(v^*\) be two common fixed points of the mappings \(T_i^m\) and \(T_j^n\).

Therefore, \(T_i^m u^* = T_j^n u^* = u^*\) and \(T_i^m v^* = T_j^n v^* = v^*\). Consider

\[
\delta_2(u^*, v^*) = \delta_2(T_i^m u^*, T_j^n v^*) = \delta_2(T_i^m u^*, T_j^n v^*)
\]

\[
\delta_2(u^*, v^*) \leq \alpha \frac{\delta_2(u^*, v^*) \cdot \delta_2(v^*, T_j^n v^*)}{\delta_2(u^*, T_j^n u^*) + \delta_2(v^*, T_j^n u^*)} + \beta \frac{\delta_2(u^*, v^*) [1 + \delta_2(u^*, T_j^n u^*) + \delta_2(v^*, T_i^m u^*)]}{1 + \delta_2(u^*, v^*)}
\]

\[
\delta_2(u^*, v^*) \leq \alpha \frac{\delta_2(u^*, v^*) \cdot \delta_2(v^*, v^*)}{\delta_2(u^*, T_j^n u^*) + \delta_2(v^*, T_i^m u^*)} + \beta \frac{\delta_2(u^*, v^*) [1 + \delta_2(u^*, u^*) + \delta_2(v^*, u^*)]}{1 + \delta_2(u^*, v^*)} + \gamma \frac{[\delta_2(u^*, u^*) \cdot \delta_2(u^*, v^*)]}{\delta_2(u^*, v^*)} + \beta \frac{\delta_2(u^*, v^*) [1 + \delta_2(v^*, u^*)]}{1 + \delta_2(u^*, v^*)}
\]

\[
\delta_2(u^*, v^*) \leq \beta \delta_2(u^*, v^*)
\]

\[
\delta_2(u^*, v^*) < \delta(u^*, v^*)
\]

Since, \(\beta < 1\). Therefore we have, \(\delta(u^*, v^*) < \delta(u^*, v^*)\). This is contradiction. Hence, \(u^*\) is a unique common fixed point of \(T_i^m\) and \(T_j^n\).
The fixed point of $T_i^m$ is a fixed point of $T_i$ and the fixed point of $T_j^n$ is fixed point of $T_j$. Therefore, $u^*$ is unique common fixed point of $T_i$ and $T_j$. Hence, $u^*$ is unique common fixed point of $P$.

This completes the proof. □

4. DISCUSSION AND THE CONCLUDING REMARKS

In this paper, we have proved the existence and uniqueness of common fixed points for two contractive type mappings in bi-b-metric space.

REFERENCES


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