HYERS-ULAM STABILITY OF FOURTH ORDER EULER’S DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work, we investigate the Hyers-Ulam stability of the fourth order Euler’s differential equations of the form

\[ t^4 y^{(iv)} + \alpha t^3 y''' + \beta t^2 y'' + \gamma t y' + \delta y = 0 \]

on any open interval \( I = (a, b) \), \( 0 < a < b \leq \infty \) or \(-\infty < a < b < 0\), where \( \alpha, \beta, \gamma \) and \( \delta \) are complex constants.

1. INTRODUCTION

The study of stability problems for various functional equations originated from a talk given by S. M. Ulam in 1940. In that talk, Ulam [15] discussed a number of important unsolved problems. Among such problems, a problem concerning the stability of functional equations: “Give conditions in order for a linear mapping near an approximately linear mapping to exist” is one of them. In 1941, Hyers [1] gave an answer to the problem.

Furthermore, the result of Hyers [1] has been generalized by Rassias [12]. After that many authors have extended the Ulam’s stability problems to other functional equations and generalized Hyer’s result in various directions (see for e.g. [2, 6, 9, 10, 11, 13, 14]). Thereafter, Ulam’s stability problem for functional equations was replaced by stability of differential equations. The differential equation

\[ a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) + h(t) = 0 \]

has Hyers-Ulam stability, if for given \( \epsilon > 0 \), \( I \) be an open interval and for any function \( f \) satisfying the differential inequality

\[ |a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) + h(t)| \leq \epsilon, \]

then there exists a solution \( f_0 \) of the above differential equation such that \( |f(t) - f_0(t)| \leq K(\epsilon) \) and \( \lim_{\epsilon \to 0} K(\epsilon) = 0 \), for \( t \in I \). If the preceding statement is also true when we replace \( \epsilon \) and \( K(\epsilon) \) by \( \phi(t) \) and \( \psi(t) \) respectively, where \( \phi, \psi : I \to [0, \infty) \) are functions not depending on \( f \) and \( f_0 \) explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability.

S. M. Jung has investigated the Hyers-Ulam stability of linear differential equations of different classes. Among his works (see for e.g [3, 4, 5, 6, 7, 8]), we are motivated by the results of [4], where he has studied the Hyers-Ulam stability of the following Euler’s differential equations:

\[ ty'(t) + ay(t) + \beta t^r x_0 = 0 \]

and also applied this result for the investigation of the Hyers-Ulam stability of the differential equation

\[ t^2 y''(t) + aty'(t) + \beta y(t) = 0, \]

where \( a, \beta \) and \( r \) are complex constants and \( x_0 \neq 0 \) is a fixed element. In [15], the authors have established the Hyers-Ulam stability of the following Euler’s differential equations

\[ t^4 y''(t) + aty'(t) + \beta y(t) + \gamma t^r x_0 = 0 \]

and

\[ t^2 y''(t) + at^2 y'(t) + \beta t^r y(t) + \gamma y(t) = 0, \]

where \( a, \beta, \gamma \) and \( r \) are complex constants with \( x_0 \neq 0 \) is a fixed element. In fact, Hyers-Ulam stability of (1.2) depends on the Hyers-Ulam stability of (1.1) for every \( x_0 \neq 0 \). In particular, we have the following results:

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\[ \text{Key words and phrases.} \quad \text{Hyers-Ulam stability, Euler’s differential equation.} \]
Theorem 1.1. [15] Let $X$ be a Complex Banach space and let $I = (a, b)$ be an open interval with either $0 < a < b \leq \infty$ or $-\infty < a < b < 0$. Assume that $\phi : I \rightarrow [0, \infty)$ is given, and that $a, \beta, \gamma, r$ are complex constants, $x_0$ is a fixed element of $X$. Furthermore, suppose a twice continuously differentiable function $f : I \rightarrow X$ satisfies
\[
\bigg\| t^{\alpha - 1} \phi(t) + \beta t y'(t) + \gamma t y(t) + \gamma t x_0 \bigg\| \leq \phi(t), \ t \in I.
\]
Let $l$ and $m$ be two numbers so that $a = 1 - l - m$ and $\beta = lm$. If $t^{r-1}, t^{r-m-1}, t^{m-1,1}, t^{m-1,1}$ and $t^{l-1}\eta(t)$ are integrable on $(a, c)$, for any $c$ with $a = c \leq b$, then there exists a unique solution $f_0 : I \rightarrow X$ of the differential equation (1.1) such that
\[
\bigg| f(t) - f_0(t) \bigg| \leq \bigg| t^{r-1} \bigg| \int_{t}^{b} u^{-l-1}\psi(u)du
\]
for all $t \in I$, where
\[
\psi(t) = \bigg| t^{m} \bigg| \int_{t}^{b} u^{-m-1}\phi(u)du.
\]

Theorem 1.2. [15] Let $X$ be a Complex Banach space and $I = (a, b)$ be an open interval such that either $0 < a < b \leq \infty$ or $-\infty < a < b < 0$. Assume that $\theta : I \rightarrow [0, \infty)$ is given along with $\alpha, \beta, \gamma$ and $r$ are complex constants and $f, m, n$ are characteristic roots of (1.2), so that $a = 3 - (l + m + n)$, $\beta = lm + mn + nl - l - m + n + 1$ and $\gamma = -(lmn)$. Let
\[
l^{m-1}, l^{m-1}, l^{m-1}, l^{m-1}, l^{m-1}, l^{m-1}, l^{m-1}, l^{m-1}, 1^{m-1}, 1^{m-1},
\]
and $t^{l-1}\eta(t)$ are integrable on $(a, c)$ with $a = c \leq b$, where
\[
\lambda(t) = \bigg| t^{m} \bigg| \int_{t}^{b} u^{-m-1}\lambda(u)du,
\]
\[
\eta(t) = \bigg| t^{m} \bigg| \int_{t}^{b} u^{-m-1}\lambda(u)du.
\]
Suppose that $f \in C^{0}(I, X)$ and satisfies
\[
\bigg| \bigg| t^{a\alpha} y''(t) + \alpha t^a y''(t) + \beta t y'(t) + \gamma y(t) \bigg| \bigg| \leq \theta(t),
\]
for all $t \in I$. Then there exists a unique solution $f_0 \in C^{0}(I, X)$ of (1.2) such that
\[
\bigg| f(t) - f_0(t) \bigg| \leq \bigg| t^{r-1} \bigg| \int_{t}^{b} u^{-l-1}\eta(u)du
\]
for all $t \in I$. Theorem 1.3. [15] Let $X$ be a Complex Banach space and let $I = (a, b)$ be an open interval such that $0 < a < b \leq \infty$ or $-\infty < a < b < 0$. Assume that a function $\phi : I \rightarrow [0, \infty)$ is given and that $h : I \rightarrow X$ is a continuous function. Furthermore, suppose a continuously differenntiable function $f : I \rightarrow X$ satisfies
\[
\bigg| f'(t) + ay(t) + h(t) \bigg| \leq \phi(t), \ t \in I.
\]
If both $t^{\alpha - 1}\phi(t)$ and $t^{\alpha - 1}h(t)$ are integrable on $(a, c)$ for any $c$ with $a = c \leq b$, then there exists a unique solution $f_0 : I \rightarrow X$ of the differential equation
\[
ty'(t) + ay(t) + h(t) = 0
\]
such that
\[
\bigg| f(t) - f_0(t) \bigg| \leq \bigg| t^{r-1} \bigg| \int_{t}^{b} u^{-l-1}\psi(u)du, \ t \in I,
\]
where
\[
f_0(t) = \left( \frac{a}{r} \right) x - t^{\alpha-1} \int_{a}^{t} u^{-l-1}h(u)du,
\]
for unique $x \in X$ and $a$ is a complex constant.

The aim of this work is to investigate the generalized Hyers-Ulam stability of the following Euler’s differential equations in the form
\[
t^{a\alpha} y''(t) + \alpha t^a y''(t) + \beta t y'(t) + \gamma y(t) + \delta t^a x_0 = 0
\]
and
\[
t^{a\alpha} y''(t) + \alpha t^a y''(t) + \beta t y'(t) + \gamma y(t) + \delta t^a x_0 = 0
\]
for all $t \in I$, where $x_0 \in X$ be a fixed element and $I = (a, b)$ with $0 < a < b \leq \infty$ or $-\infty < a < b < 0$, then there exists a unique solution $f_0 \in C^{0}(I, X)$ of (1.5) such that
\[
\bigg| f(t) - f_0(t) \bigg| \leq \bigg| t^{r-1} \bigg| \int_{t}^{b} u^{-l-1}\theta(u)du
\]
for all $t \in I$, where
\[
\theta(t) = \bigg| t^{r-1} \bigg| \int_{t}^{b} u^{-l-1}\theta(u)du,
\]
\[
\psi(t) = \bigg| t^{r-1} \bigg| \int_{t}^{b} u^{-l-1}\phi(u)du,
\]
and $l, m, n$ are the characteristic roots of (1.3) such that $\alpha = 3 - (l + m + n)$, $\beta = lm + mn + nl - l - m - n + 1$ and $\gamma = -(lmn)$. Also, we apply this result to investigate the Hyers-Ulam stability of (1.4). Throughout this work, we let $I = (a, b)$ either $0 < a < b \leq \infty$ or $-\infty < a < b < 0$. 

2. Hyers-Ulam Stability

Let \( l, m \) and \( n \) be the characteristic roots of (1.3) such that \( \alpha = 3 - (l + m + n), \beta = lm + mn + nl - l - m - n + 1 \) and \( \gamma = -(lmn) \) with \( g(r) = r^3 + (\alpha - 3)r^2 + (\beta - \alpha + 2)r + \gamma. \) For fixed \( x_0 \neq 0, \) the possible solutions of (1.3) in the class of real valued functions defined on \( I \) are given by

(I) When \( g(r) \neq 0, \)
\[
y(t) = \begin{cases} 
c_1 t^l + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|}{g(r)}, \\
for \ l \neq m \neq n, 
\end{cases}
\]
\[
y(t) = \begin{cases} 
c_1 t^l + c_2 t^m - \frac{\delta x_0 t^l \ln|t|}{g(r)}, \\
for \ l = m \neq n, 
\end{cases}
\]
\[
y(t) = \begin{cases} 
c_1 + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|}{g(r)}, \\
for \ l = n \neq m, 
\end{cases}
\]
\[
y(t) = \begin{cases} 
c_1 + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|^2}{g(r)}, \\
for \ l = m = n. 
\end{cases}
\]

(II) When \( g(r) = 0 \neq g'(r), \)
\[
y(t) = \begin{cases} 
c_1 t^l + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|}{g(r)}, \\
for \ l \neq m \neq n, 
\end{cases}
\]
\[
y(t) = \begin{cases} 
c_1 t^l + c_2 t^m - \frac{\delta x_0 t^l \ln|t|}{g(r)}, \\
for \ l = m \neq n, 
\end{cases}
\]
\[
y(t) = \begin{cases} 
c_1 + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|^2}{g(r)}, \\
for \ l = n \neq m, 
\end{cases}
\]
\[
y(t) = \begin{cases} 
c_1 + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|^2}{g(r)}, \\
for \ l = m = n. 
\end{cases}
\]

Remark 2.1. Indeed, \( g'(r) = 3r^3 + 2(\alpha - 3)r + (\beta - \alpha + 2). \) If \( g(r) = 0, \) then either \( r - l = 0 \) or \( r - m = 0 \) or \( r - n = 0. \) Therefore, \( r - l = 0 \) and \( g'(r) \neq 0 \) implies that \( 3r^3 + 2(\alpha - 3)r + (\beta - \alpha + 2) = (r - m)(r - n) \neq 0. \) So \( r - m \neq 0 \) and \( r - n \neq 0. \) Similarly, when \( r - m = 0 \) and \( g'(r) \neq 0 \) implies \( r - l \neq 0 \) and \( r - n \neq 0 \) and when \( r - n = 0 \) and \( g'(r) \neq 0 \) implies \( r - l \neq 0 \) and \( r - m \neq 0. \) Hence the first solution for \( l \neq m \neq n \) could be any one of the following:

\[
\begin{align*}
c_1 t^l + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|}{(r - m)(r - n)} \\
r - l = 0, r - m \neq 0, r - n \neq 0,
\end{align*}
\]
\[
\begin{align*}
c_1 t^l + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|}{(r - l)(r - m)} \\
r - n = 0, r - l \neq 0, r - m \neq 0,
\end{align*}
\]
\[
\begin{align*}
c_1 t^l + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|}{(r - l)(r - n)} \\
r - m = 0, r - l \neq 0, r - n \neq 0.
\end{align*}
\]

(III) When \( g(r) = 0 \neq g'(r), \) but \( g''(r) \neq 0, \)
\[
y(t) = \begin{cases} 
(c_1 + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|^2}{g(r)}), \\
for \ l = m = r, r - n \neq 0, 
\end{cases}
\]
\[
y(t) = \begin{cases} 
(c_1 t^l + (c_2 + c_3 t^m) \ln|t| - \frac{\delta x_0 t^l \ln|t|}{g(r)}), \\
for \ m = n = r, r - l \neq 0, 
\end{cases}
\]
\[
y(t) = \begin{cases} 
(c_1 + c_2 t^m + c_3 t^n - \frac{\delta x_0 t^l \ln|t|^2}{g(r)}), \\
for \ l = n = r, r - m \neq 0. 
\end{cases}
\]

(IV) When \( g(r) = 0 \neq g'(r) = g''(r), \)
\[
y(t) = \begin{cases} 
(c_1 + c_2 t^m + c_3 t^n + (\ln|t|)^2 \frac{\delta x_0 t^l}{g^3(r)}), \\
for \ l = n = r, r - m \neq 0. 
\end{cases}
\]

where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants.

Theorem 2.1. Let \( X \) be a Complex Banach space. Assume that a function \( \phi : I \rightarrow [0, \infty) \) is given, that \( \alpha, \beta, \gamma, \delta, \tau \) are complex constants and that \( x_0 \) is a fixed element of \( X \). Furthermore, suppose a thrice continuously differentiable function \( f : I \rightarrow X \) satisfies the differential inequality (1.5). If \( t^{r-1}, t^{m-1}, t^{n-1}, t^{m-n-1}, t^{m-1} - t^{m-n-1} \ln |\frac{t}{2}|, t^{r-n-1} - t^{m-n-1} \ln |\frac{t}{2}|, t^{r-n-1} - t^{m-n-1} \ln |\frac{t}{2}|, t^{r-n-1} t^{m-n-1} \ln |\frac{t}{2}|, t^{r-n-1} t^{m-n-1} \ln |\frac{t}{2}|, t^{r-n-1} \phi(t), t^{r-n-1} \psi(t) \) and \( t^{m-n-1} \tau(t) \) are integrable on \( (a, c), \) for any \( c \) with \( a < c \leq b, \) then there exists a unique solution \( f_0 \in C^3(I, X) \) of (1.3) such that (1.6) holds for any \( t \in I, \) where \( l, m \) and \( n \) are the roots of \( g(r) = 0. \)

Proof. To prove the theorem it is sufficient to consider the following cases:

(i) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(ii) \( (r - l)(r - m)(r - n) \neq 0, l \neq m = n; \)

(iii) \( (r - l)(r - m)(r - n) \neq 0, l = m \neq n; \)

(iv) \( (r - l)(r - m)(r - n) \neq 0, l = m = n; \)

(v) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(vi) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(vii) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(viii) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(ix) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(x) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(xi) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(xii) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(xiii) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(xiv) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

(xv) \( (r - l)(r - m)(r - n) \neq 0, l \neq m \neq n; \)

Case-(i) Suppose that \( X \) is a Complex Banach space and a thrice continuously differentiable function \( f : I \rightarrow X \) is satisfying the differential inequality (1.3). Let \( l, m \) and \( n \) be the roots of \( g(u) = u^3 + (\alpha - 3)u^2 + (\beta - \alpha + 2)u + \gamma = 0. \) Define \( h : I \rightarrow X \)
such that \( h(t) = tf(t) - nf(t) \). Then we have
\[
\begin{align*}
\| t^2 h''(t) + (1 - l - m)t h(t) + \ln h(t) + \delta t^r x_0 \| &= \| t^3 f''(t) + \alpha t^2 f'(t) + \beta t f(t) + \gamma f(t) + \delta t^r x_0 \|
\leq \phi(t).
\end{align*}
\]
Hence by Theorem 1.1 and then using (1.7), it follows that there exists a unique solution \( h_0 : I \to X \) of the differential equation
\[
t^2 y''(t) + (1 - l - m)y'(t) + \ln y(t) + \delta t^r x_0 = 0
\]
such that
\[
(2.1) \quad \| h(t) - h_0(t) \| \leq \| f(t) - n f(t) - h_0(t) \| \leq \phi(t).
\]
where \( h_0(t) = k_1 t^l + k_2 r^m + k_3 t^r \) with
\[
\begin{align*}
k_1 &= \left\{ \frac{x_1}{a^m} - \frac{\delta x_0 a^r}{(r-l)(m-l) - \delta^r(m-l)} \right\} \, t^n, \\
k_2 &= \left\{ \frac{x_2}{a^m} - \frac{\delta x_0 a^r}{(r-m)(l-m)} \right\}, \\
k_3 &= -\frac{\delta x_0}{(r-l)(r-m)}.
\end{align*}
\]
Consequently, (2.1) becomes
\[
\| f(t) - f_0(t) \| \leq \| v^n \| \int_a^b u^{-n-1} v(t) \mathrm{d}u, \quad t \in I
\]
provided \( t^{-n-1} \theta(t) \) and \( t^{-n-1} h_0(t) \) are integrable on \((a, c)\) for any \( c \) with \( a < c < b \). As \( t^{l-n-1}, t^{m-n-1}, t^{r-n-1}, \ln \frac{t}{a} \), \( t^{n-1} \ln \frac{x}{l} \) and \( t^{l-n-1} \ln \frac{x}{l} \) are integrable, then so also \( t^{-n-1} h_0(t) \). According to Theorem 1.3, \( f_0(t) \) is given by
\[
(2.2) \quad f_0(t) = \left( \frac{t}{a} \right)^n \tilde{x} + t^n \int_a^t u^{-n-1} h_0(u) \mathrm{d}u,
\]
where \( \tilde{x} \) is a limit point in \( X \). It is easy to verify that
\[
\int_a^t u^{-n-1} h_0(u) \mathrm{d}u = \frac{k_1}{l-n} \left( t^{l-n} - d^{l-n} \right) + \frac{k_2}{m-n} \left( t^{m-n} - a^{m-n} \right) + \frac{k_3}{r-n} \left( t^{r-n} - a^{r-n} \right).
\]
As a result,
\[
f_0(t) = \frac{k_1}{l-n} t^l + \frac{k_2}{m-n} t^m + \frac{k_3}{r-n} t^r \delta x_0 t^r \frac{\gamma}{\delta x_0 t^r}.
\]
\[\text{Case-(ii)}\] Proceeding as in Case (i), we can obtain
\[
\int_a^t u^{-n-1} h_0(u) \mathrm{d}u = \frac{k_1}{l-n} \left( t^{l-n} - d^{l-n} \right) + k_2 \ln \frac{t}{a} + \frac{k_3}{r-n} \left( t^{r-n} - a^{r-n} \right).
\]
Consequently,
\[
f_0(t) = \frac{k_1}{l-m} t^l + \frac{\delta x_0 a^r}{a^m} \left( t^{l-n} - d^{l-n} \right) + k_2 \ln \frac{t}{a} + \frac{k_3}{r-n} \left( t^{r-n} - a^{r-n} \right),
\]
where \( k_1, k_2, k_3 \) are same as in Case (i).
\[\text{Case-(iii)}\] Proceeding as before with
\[
h_0(t) = \{ k_4 + k_5 \ln \frac{t}{a} \} \, t^l + k_6 t^r
\]
and
\[
k_4 = \frac{x_2}{a^m} \left( \frac{\delta x_0 a^r}{(r-l)^2} \right),
\]
k_5 = \frac{x_2}{a^m} \left( \frac{\delta x_0 a^r}{l-n} \right),
\]
k_6 = \frac{-\delta x_0}{(r-l)^2}.
\]
it is easy to verify that
\[
\int_a^t u^{-n-1} h_0(u) \mathrm{d}u = \frac{k_3}{l-n} \left( t^{l-n} - d^{l-n} \right) + \frac{k_5}{l-n} \left( \ln \frac{t}{a} - \frac{1}{l-n} \right) \, t^{l-n} + \frac{k_1}{l-n} \left( t^{l-n} - \frac{d^{l-n}}{(l-n)^2} \right) + \frac{k_4}{l-n} \left( t^{l-n} - \frac{d^{l-n}}{(l-n)^2} \right),
\]
\[
\text{Hence, from (2.2)} \quad f_0(t) = \left[ \frac{k_4}{l-n} - \frac{k_6}{l-n} \right] \, t^l \ln \frac{t}{a} + \frac{k_1}{l-n} \left( t^{l-n} - d^{l-n} \right) + \frac{k_2}{m-n} \left( t^{m-n} - a^{m-n} \right) + \frac{k_5}{l-n} \left( t^{l-n} - \frac{d^{l-n}}{(l-n)^2} \right) - \frac{k_3}{r-n} \left( t^{r-n} - a^{r-n} \right).
\]
\[\text{Case-(iv)}\] We proceed as in Case (i) and it is easy to see that
\[
\int_a^t u^{-n-1} h_0(u) \mathrm{d}u = k_1 \ln \frac{t}{a} + \frac{k_2}{m-n} \left( t^{m-n} - a^{m-n} \right) + \frac{k_3}{r-n} \left( t^{r-n} - a^{r-n} \right).
\]
So from (2.2), it happens that
\[
f_0(t) = \frac{k_2}{m-n} t^m + \frac{k_3}{r-n} t^r \frac{\delta x_0 t^r}{g(r)}.
\]
Case-(v) In this case we have that
\[ h_0(t) = \left( k_1 + k_0 \ln \left| \frac{t}{a} \right| \right) t^l + k_0 t^r, \]
where \( k_4, k_5, \) and \( k_6 \) are same as in Case (iii). Therefore,
\[ \int_a^t u^{-n-1} h_0(u) du = k_4 \ln \left| \frac{t}{a} \right| + \]
\[ + \frac{k_5}{2} \left( \ln \left| \frac{t}{a} \right| \right)^2 + \frac{k_6}{r-n} \left( t^{r-n} - a^{r-n} \right) \]
implies from (2.2) that
\[ f_0(t) = \left( \frac{x}{a^n} - \frac{k_0 t^{r-n}}{(r-n)} + \frac{k_4}{a^n} \ln \left| \frac{t}{a} \right| + \frac{k_5}{r-n} \left( \ln \left| \frac{t}{a} \right| \right)^2 \right) \]
\[ \left( \frac{x}{a^n} - \frac{k_0 t^{r-n}}{(r-n)} + \frac{k_4}{a^n} \ln \left| \frac{t}{a} \right| + \frac{k_5}{r-n} \left( \ln \left| \frac{t}{a} \right| \right)^2 \right) t^l \]
\[ - \frac{\delta x_0 t^r}{(r-n)(r-n)} \ln \left| \frac{t}{a} \right|. \]

Case-(vi) In this case, we notice that
\[ h_0(t) = k_7 t^l + k_5 t^m + k_0 t^r \ln \left| \frac{t}{a} \right|, \]
where
\[ k_7 = \left( \frac{x}{a^n} + \frac{\delta x_0}{a^n} \right) \]
\[ + \frac{k_9}{a^n} \left( t^m - a^m \right) + \frac{k_8}{a^n} \left( t^l - a^l \right) \]
\[ k_8 = \left( \frac{x t^l}{a^n} - \frac{\delta x_0 a^{l-n}}{a^n} \right) \]
\[ k_9 = - \frac{\delta x_0}{a^n}, \]

and
\[ \int_a^t u^{-n-1} h_0(u) du = \frac{k_7}{a^n} \left( t^m - a^m \right) + \frac{k_8}{a^n} \left( t^l - a^l \right) \]
\[ + \frac{k_9}{a^n} \left( t^{m-n} - a^{m-n} \right) \]
\[ + \frac{k_{10}}{m-n} \left( t^{n-m} - a^{n-m} \right) \]
\[ + \frac{k_{11}}{r-n} \left( t^{r-n} - a^{r-n} \right) \]
\[ + \frac{k_{12}}{(r-n)^2} \left( \ln \left| \frac{t}{a} \right| \right) \]
\[ - \frac{\delta x_0 t^r}{(r-n)(r-n)} \ln \left| \frac{t}{a} \right|. \]

Hence from (2.2), it follows that
\[ f_0(t) = \left( \frac{k_7}{l-n} - \frac{k_9}{a^n}, \right) \]
\[ + \frac{k_8}{m-n} t^m + \frac{k_9}{m-n} \]
\[ + \frac{\delta x_0}{m-n} \]
\[ - \frac{\delta x_0 t^r}{(r-n)(r-n)} \ln \left| \frac{t}{a} \right|. \]

Case-(vii) Here, we have
\[ h_0(t) = k_1 t^l + k_5 t^m + k_9 t^r, \]

and
\[ \int_a^t u^{-n-1} h_0(u) du = \frac{k_1}{l-n} \left( t^l - a^l \right) + \]
\[ + \frac{k_2}{m-n} \left( t^m - a^m \right) + \frac{k_3}{a^n} \ln \left| \frac{t}{a} \right|, \]

where \( k_1, k_2, \) and \( k_3 \) are same as in Case-(i). Applying this in (2.2), we obtain
\[ f_0(t) = \left( \frac{k_1}{l-n} \right) t^l + \frac{k_2}{m-n} t^m + \]
\[ + \frac{\delta x_0}{a^n} \left( t^m - a^m \right) \]
\[ - \frac{\delta x_0 t^r}{(r-n)(r-n)} \ln \left| \frac{t}{a} \right|. \]

Case-(viii) For this case, \( h_0(t) \) becomes
\[ h_0(t) = k_{10} t^l + k_{11} t^m + k_{12} t^r \ln \left| \frac{t}{a} \right|, \]
where
\[ k_{10} = \left( \frac{x}{a^n} - \frac{k_0 t^{r-n}}{(r-n)} \right) \]
\[ + \frac{k_{11}}{m-n} \left( t^{m-n} - a^{m-n} \right) \]
\[ + \frac{k_{12}}{r-n} \left( t^{r-n} - a^{r-n} \right) \]
\[ - \frac{\delta x_0 t^r}{(r-n)(r-n)} \ln \left| \frac{t}{a} \right|. \]

Using
\[ \int_a^t u^{-n-1} h_0(u) du = \frac{k_{10}}{l-n} \left( t^l - a^l \right) + \]
\[ + \frac{k_{11}}{m-n} \left( t^m - a^m \right) \]
\[ + \frac{k_{12}}{r-n} \left( t^r - a^r \right) \]
\[ - \frac{\delta x_0 t^r}{(r-n)(r-n)} \ln \left| \frac{t}{a} \right|. \]

in (2.2), we find
\[ f_0(t) = \left( \frac{k_{10}}{l-n} \right) t^l + \left( \frac{k_{11}}{m-n} \right) t^m + \]
\[ + \frac{\delta x_0}{a^n} \left( t^m - a^m \right) \]
\[ - \frac{\delta x_0 t^r}{(r-n)(r-n)} \ln \left| \frac{t}{a} \right|. \]

Case-(ix) Here \( h_0(t) = k_7 t^l + k_9 t^m + k_9 t^r \ln \left| \frac{t}{a} \right| \), where \( k_7, k_9, \) and \( k_9 \) are same as in Case-(vi). Using the fact
\[ \int_a^t u^{-n-1} h_0(u) du = \frac{k_7}{l-n} \left( t^l - a^l \right) + \]
\[ + \frac{k_9}{r-n} \left( t^{r-n} - a^{r-n} \right) \]
\[ - \frac{\delta x_0 t^r}{(r-n)(r-n)} \ln \left| \frac{t}{a} \right|. \]

in (2.2), it follows that
\[ f_0(t) = \left( \frac{k_7}{l-m} \right) t^l + \]
\[ + \frac{\delta x_0}{a^n} \left( t^m - a^m \right) \]
\[ - \frac{\delta x_0 t^r}{(r-m)(r-m)} \ln \left| \frac{t}{a} \right|. \]

Case-(x) In this case, \( h_0(t) = \left\{ k_4 + k_6 \ln \left| \frac{t}{a} \right| \right\} t^l + k_9 t^r \), where \( k_4, k_6, \) and \( k_9 \) are same as in Case-(iii).
Now, it is easy to verify that
\[
\int_a^t v^{-n-1} h_0(v) dv = \frac{k_4}{l - n} \left( \frac{t^{l - n}}{l - n} - a^{l - n} \right) + \\
+ \frac{k_5}{l - n} \left[ \frac{t^{l - n}}{l - n} \ln \frac{t}{a} - \frac{a^{l - n}}{l - n} \right] + k_6 \ln \frac{t}{a}.
\]
Hence, from (2.2) we obtain
\[
f_0(t) = \left[ \left\{ \frac{k_4}{l - n} \right\} \left( \frac{t^{l - n}}{l - n} - a^{l - n} \right) + \frac{k_5}{l - n} \ln \frac{t}{a} \right] t^l + \\
+ \left[ \frac{a}{a^n} - \frac{k_4 a^{l - n}}{l - n} + \frac{k_5 a^{l - n}}{(l - n)^2} \right] t^m - \\
- \frac{\delta x_0 t^r}{2(r - l)^2} \ln \frac{t}{a}.
\]

Case (xi) In this case, \( h_0(t) = k_{10} t^l + k_{11} t^m + k_{12} t^r \ln \frac{t}{a} \), where \( k_{10}, k_{11} \) and \( k_{12} \) are same as in Case (viii). Clearly,
\[
\int_a^t v^{-n-1} h_0(v) dv = k_{10} \ln \frac{t}{a} + \\
+ \frac{k_{11}}{m - n} (t^{m-n} - a^{m-n}) + \\
+ \frac{k_{12}}{r - n} \left[ \frac{t^{r-n}}{r - n} - \frac{a^{r-n}}{r - n} \right] t^m - \\
- \frac{\delta x_0 t^r}{2(r - l)^2} \ln \frac{t}{a}.
\]
and (2.2) reduces to
\[
f_0(t) = \left[ \frac{a}{a^n} - \frac{k_{11}}{m - n} + \frac{k_{12}}{r - n} \right] t^m - \frac{\delta x_0 t^r}{2(r - l)^2} \ln \frac{t}{a}.
\]

Case (xii) In this case we obtain
\[
h_0(t) = \left[ \frac{x_6}{a^l} + \frac{x}{a^l} \ln \frac{t}{a} \right] t^l - \frac{\delta x_0 t^r}{2} \left( \ln \frac{t}{a} \right)^3
\]
and thus
\[
\int_a^t v^{-n-1} h_0(v) dv = \frac{x_6}{a^l(t-n)} \left[ t^{l-n} - a^{l-n} \right] + \\
+ \frac{x}{a^l(t-n)} \left[ \frac{t^{l-n}}{l-n} - \frac{a^{l-n}}{l-n} \right] - \frac{\delta x_0}{a^n} \times \\
\times \left[ t^{l-n} \left( \ln \frac{t}{a} \right)^3 - \frac{x_6}{a^n(t-n)^2} \ln \frac{t}{a} + \frac{x}{a^n(t-n)^2} - \frac{\delta x_0}{a^n} \left( \ln \frac{t}{a} \right)^3 \right]
\]
which on applying in (2.2), we get
\[
f_0(t) = \left[ \frac{x_6}{a^l(t-n)} - \frac{x}{a^l(t-n)^2} - \frac{\delta x_0}{a^n} \times \\
+ \left[ \frac{x}{a^l(t-n)^2} - \frac{\delta x_0}{a^n(t-n)^2} \ln \frac{t}{a} \right] t^l \right] + \\
\left[ \frac{a}{a^n} - \frac{x_6}{a^n(t-n)} + \frac{x}{a^n(t-n)^2} + \frac{\delta x_0 a^{l-n}}{(l-n)^2} \right] t^m - \\
- \frac{\delta x_0 t^r}{2(r - l)^2} \left( \ln \frac{t}{a} \right)^3.
\]

Case (xiii) Here \( h_0(t) = k_{10} t^l + k_{11} t^m + k_{12} t^r \ln \frac{t}{a} \), where \( k_{10}, k_{11} \) and \( k_{12} \) are same as in Case (viii) and
\[
\int_a^t v^{-n-1} h_0(v) dv = \frac{k_{10}}{l - n} \left( t^{l-n} - a^{l-n} \right) + \\
+ \frac{k_{11}}{l - n} \ln \frac{t}{a} + \frac{k_{12}}{2(r - l)^2} \left( \ln \frac{t}{a} \right)^3.
\]
Applying this in (2.2), it follows that
\[
f_0(t) = \frac{k_{10}}{l - n} t^l + \left[ \frac{\delta a}{a^n} - \frac{k_{11}}{l - n} \ln \frac{t}{a} \right] t^m - \\
- \frac{\delta x_0 t^r}{2(r - l)} \left( \ln \frac{t}{a} \right)^3.
\]

Case (xiv) In this case,
\[
h_0(t) = k_7 t^l + k_8 t^m + k_9 t^r \ln \frac{t}{a},
\]
where \( k_7, k_8 \) and \( k_9 \) are same as in Case (vi) and
\[
\int_a^t v^{-n-1} h_0(v) dv = k_7 \ln \frac{t}{a} + \\
+ \frac{k_8}{m - n} (t^{m-n} - a^{m-n}) + \frac{k_9}{2} \left( \ln \frac{t}{a} \right)^3.
\]
Ultimately, (2.2) becomes
\[
f_0(t) = \left( \frac{k_8}{m - n} \right) t^m + \\
+ \left[ \frac{\delta a}{a^n} - \frac{k_8}{m - n} \ln \frac{t}{a} \right] t^m - \\
- \frac{\delta x_0 t^r}{2(r - m)} \left( \ln \frac{t}{a} \right)^3.
\]

Case (xv) For this case,
\[
h_0(t) = \left[ \frac{x_6}{a^l} + \frac{x}{a^l} \ln \frac{t}{a} \right] t^l - \frac{\delta x_0 t^r}{2} \left( \ln \frac{t}{a} \right)^3
\]
and hence
\[
\int_a^t v^{-n-1} h_0(v) dv = \frac{x_6}{a^l(t-n)} \left[ t^{l-n} - a^{l-n} \right] + \\
+ \frac{x}{a^l(t-n)} \left[ \frac{t^{l-n}}{l-n} - \frac{a^{l-n}}{l-n} \right] - \frac{\delta x_0}{a^n} \times \\
\times \left[ t^{l-n} \left( \ln \frac{t}{a} \right)^3 - \frac{x_6}{a^n(t-n)^2} \ln \frac{t}{a} + \frac{x}{a^n(t-n)^2} - \frac{\delta x_0}{a^n} \left( \ln \frac{t}{a} \right)^3 \right]
\]
which on applying in (2.2), we get
\[
f_0(t) = \left[ \frac{x_6}{a^l(t-n)} - \frac{x}{a^l(t-n)^2} - \frac{\delta x_0}{a^n} \times \\
+ \left[ \frac{x}{a^l(t-n)^2} - \frac{\delta x_0}{a^n(t-n)^2} \ln \frac{t}{a} \right] t^l \right] + \\
\left[ \frac{a}{a^n} - \frac{x_6}{a^n(t-n)} + \frac{x}{a^n(t-n)^2} + \frac{\delta x_0 a^{l-n}}{(l-n)^2} \right] t^m - \\
- \frac{\delta x_0 t^r}{2(r - l)^2} \left( \ln \frac{t}{a} \right)^3.
\]
Using this in (2.2), \( f_0(t) \) can be obtained as
\[
f_0(t) = \left[ \frac{\delta a}{a^n} + \frac{x_6}{a^l(t-n)} + \frac{x}{a^l(t-n)^2} + \frac{\delta x_0 a^{l-n}}{(l-n)^2} \right] t^l - \\
- \frac{\delta x_0 t^r}{6} \left( \ln \frac{t}{a} \right)^3.
\]
Here \( \delta, x, x_1, x_2, x_3, x_4 \) and \( x_6 \) are all limit points in \( X \). This completes the proof of the theorem. □
3. Main Results

In this section, we investigate the Hyers-Ulam stability of (1.4) on I. Assume that I, m, n, and p are the characteristic roots (1.4) such that

\[ \alpha = \{6 - (l + m + n + p)\}, \]

\[ \beta = \{7 - 3(l + m + n + p) + (lm + mn + nl + np + lp + mp)\}, \]

\[ \gamma = \{1 - (l + m + n + p) + (lm + mn + nl + mp + lp + np) - (lmm + lnp + lmp)\}, \]

\[ \delta = lmn, \alpha_1 = \{3 - (l + m + n)\}, \]

\[ \beta_1 = \{lm + mn + nl - l - m - n + 1\} \] and 

\[ \gamma_1 = -(lmm). \]

**Theorem 3.1.** Let \( X \) be a Complex Banach space. Assume that a function \( \eta : I \to [0, \infty) \) is given. Furthermore assume that \( t^{p-1}, \ t^{m-1}, \ t^{p-1} - n, \ t^{m-1} - n, \ t^{m-1} - n^2, \ t^{n-1}, \ t^{n-1} - n \), \( t^{n-1} \eta(t) \), \( t^{n-1} \phi(t) \), and \( t^{n-1} \theta(t) \) are integrable over the interval \((a, c)\) with \( a \leq c \), where

\[ \phi(t) = \int_t^b u^{-p-1} \eta(u)du, \]

\[ \psi(t) = \int_t^b u^{-m-1} \phi(u)du, \]

\[ \theta(t) = \int_t^b u^{-1} \psi(u)du. \]

Suppose that \( f \in C^4(I, X) \) satisfies the differential inequality

\[ \left\| t^4 y^{(iv)}(t) + \alpha t^3 y^{(iv)}(t) + \beta t^2 y^{(iv)}(t) + \gamma ty(t) + \delta y(t) \right\| \leq \eta(t), \]

for all \( t \in I \). Then there exists a unique solution \( f_0 \in C^4(I, X) \) of (1.4) such that

\[ \left\| f(t) - f_0(t) \right\| \leq \int_t^b u^{-n-1} \theta(u)du. \]

**Proof.** Let \( X \) be a Complex Banach space and \( f : I \to X \) such that (3.1) hold, for \( t \in I \). Define \( s : I \to X \) such that

\[ s(t) = t^3 f''''(t) + \alpha t^3 f''''(t) + \beta t f'(t) + \gamma f(t). \]

Indeed,

\[ \left\| t^3 s(t) - ps(t) \right\| = \left\| t^4 f^{(iv)}(t) + \alpha t^3 f^{(iv)}(t) + \beta t^2 f^{(iv)}(t) + \gamma t f'(t) + \delta f(t) \right\| \leq \eta(t). \]

From Theorem 1.3, it follows that there exists a unique solution \( s_0 : I \to X \) of the differential equation

\[ t^3 s(t) - ps(t) = 0 \]

such that

\[ \left\| s(t) - s_0(t) \right\| \leq \int_t^b u^{-p-1} \eta(u)du, \]

where \( s_0(t) = \left(\frac{1}{t}\right)^p x_0 \) and \( x_0 \in X \) is a limit point. If we denote

\[ \phi(t) = \int_t^b u^{-p-1} \eta(u)du, \]

then clearly \( \phi : I \to [0, \infty) \) and

\[ \left\| s(t) - s_0(t) \right\| \leq \phi(t). \]

Therefore, from (3.2) and (3.3) we get

\[ \left\| f(t) - f_0(t) \right\| \leq \int_t^b u^{-n-1} \theta(u)du, \]

where

\[ \theta(t) = \int_t^b u^{-n-1} \psi(u)du. \]

Here, \( f_0(t) \) can be made any of the following cases

\[ i) \quad f_0(t) = \frac{e_1}{l - n} t^d + \frac{e_2}{m - n} t^{a_n} + \frac{e_3}{p - n} t^{a_p}, \]

where

\[ e_1 = \left\{ \frac{x_1}{a^2} + \frac{x_0}{a(p - l)(m - l)} - \frac{x}{a^2(l - m)} \right\}, \]

\[ e_2 = \left\{ \frac{x}{a^m(l - m)} + \frac{x_0}{a^m(m - l)(l - m)} \right\} \] and 

\[ e_3 = \frac{x_0}{a^p(p - l)(p - m)} \] and

\[ ii) \quad f_0(t) = \left(\frac{e_1}{l - n}\right) t^d + \frac{e_2}{m - n} t^{a_n} + \frac{e_3}{p - n} t^{a_p}, \]

where

\[ e_1 = \left\{ \frac{x_1}{a^2} + \frac{x_0}{a(p - l)(m - l)} - \frac{x}{a^2(l - m)} \right\}, \]

\[ e_2 = \left\{ \frac{x}{a^m(l - m)} + \frac{x_0}{a^m(m - l)(l - m)} \right\} \] and 

\[ e_3 = \frac{x_0}{a^p(p - l)(p - m)^2}. \]
(iii) \( f_0(t) = \left\{ \frac{e_4}{l-n} - \frac{e_5}{(l-n)^2} + \frac{e_6}{l-n} \left| \frac{t}{a} \right| \right\} t^n + \left( \frac{x}{a^n} - \frac{e_4 e_m}{l-n} + \frac{e_6 a^p}{p-n} \right) t^n + \frac{x_0}{a^p(p-m)(p-n)^2} t^p, \)

where

\[
e_4 = \left\{ \frac{x_2}{a^2} - \frac{x_0}{a^2(p-l)} \right\},
\[
e_5 = \left\{ \frac{x}{a^2} - \frac{x_0}{a^2(p-l)} \right\} \text{ and } d
\]

\[
e_6 = \frac{x_0}{a^p(p-l)^2},
\]

(iv) \( f_0(t) = \left( \frac{e_4}{m-n} \right) t^n + \left( \frac{x}{a^n} - \frac{e_4 a^p}{m-n} + e_4 \ln \left| \frac{t}{a} \right| \right) t^n + \frac{x_0}{a^p(p-m)(p-n)^2} t^p. \)

(v) \( f_0(t) = \)

\[
\left\{ \frac{e_4}{l-n} - \frac{e_5}{l-n} + \frac{e_6}{2} \left( \ln \left| \frac{t}{a} \right| \right)^2 \right\} t^n + \frac{x_0}{a^p(p-l)^2} t^p.
\]

(vi) \( f_0(t) = \frac{e_7}{l-n} + \frac{e_8}{m-n} t^n + \left( \frac{x}{a^2} - \frac{e_7 a^l}{l-n} + \frac{e_8 a^m}{m-n} \right) t^n + \frac{x_0}{a^p(p-l)^2} t^p. \)

where

\[
e_7 = \left\{ \frac{x_2}{a^2} - \frac{x_0}{a^2(l-m)^2} + \frac{x}{a^2(l-m)} \right\},
\]

\[
e_8 = \left\{ \frac{x}{a^2(l-m)^3} - \frac{x_0}{a^2(l-m)^2} \right\} \text{ and } d
\]

\[
e_9 = \frac{x_0}{a^p(a^p(p-m))},
\]

(vii) \( f_0(t) = \frac{e_1}{l-n} + \frac{e_2}{m-n} t^n + \left( \frac{x}{a^2} - \frac{e_1 a^l}{l-n} + \frac{e_2 a^m}{m-n} \right) t^n + \frac{x_0}{a^p(p-l)(p-m)} t^p \ln \left| \frac{t}{a} \right|. \)

(viii) \( f_0(t) = \left( \frac{e_1}{l-n} \right) t^n + \left( \frac{e_11}{m-n} \right) t^n + \left( \frac{e_12}{p-n} \right) t^n + \frac{x_0}{a^p(p-l)(p-m)^2} t^p. \)

(ix) \( f_0(t) = \left( \frac{e_11}{m-n} \right) t^n + \left( \frac{e_12}{p-n} \right) t^n + \left( \frac{e_10}{a^n} \right) t^n + \frac{x_0}{a^p(p-l)^2} \left( \ln \left| \frac{t}{a} \right| \right)^2. \)
\[(x\nu)_{f_0}(t) = \left(\frac{e_a}{m - n}\right) t^m + \left\{ \frac{\xi}{a^n} - e_8 \frac{a^m - n}{m - n} + e_7 \ln \left(\frac{t}{a}\right) \right\} t^n + \frac{x_0}{2a^p(b - m)} \left( \ln \left(\frac{t}{a}\right) \right)^2,\]

and
\[(x\nu)_{f_0}(t) = \left\{ \frac{\xi}{a^n} + \frac{x_0}{a^n} \ln \left(\frac{t}{a}\right) + \frac{x_0}{2a^p(b - m)} \left( \ln \left(\frac{t}{a}\right) \right)^2 + \frac{x_0}{6a^n} \left( \ln \left(\frac{t}{a}\right) \right)^3 \right\} t^q,\]

where \(x, \xi, x_0, x_1, x_2, x_3, x_4\) and \(x_5\) are limit points in \(X\). These are the unique solutions of all possible cases. In fact, all these are the possible solutions of (1.4). Hence the theorem is proved.

**Example 3.1.** Let \(X = \mathbb{R}\) be a Banach space and \(I = (a, \infty), a > 0\). Consider the Euler's equation
\[(3.5)\]
\[t^\gamma y''(t) + 2t^{\gamma - 1} y'(t) - 4t^{\gamma - 2} y(t) = 0\]

If we compare (3.5) with (1.4), then \(a = 0, \beta = 2, \gamma = -4\) and \(d = 4\), and \(l = 1, m = 1, n = 2, p = 2\) are the characteristic roots of (3.5). Let \(f : I \to X\) satisfy the differential inequality
\[
\left\| t^\epsilon f^{(\nu)}(t) + 2t^\epsilon y''(t) - 4t^\epsilon y'(t) + 4t^\epsilon f(t) \right\| \leq \epsilon
\]
for any \(\epsilon > 0\) and for any \(t \in I\). Then, by Theorem 3.1, there exists a unique solution \(f_0 : I \to X\) such that
\[
\left\| f(t) - f_0(t) \right\| \leq \left\| t^\epsilon \right\| \left( \int_a^b u^{-\epsilon} \vartheta(u) du \right).
\]

where \(I = (a, b)\) and
\[
\varphi(t) = \left\| t^\epsilon \right\| \left( \int_a^b u^{-\epsilon} e^u du \right) = e^2 \left( \frac{t}{b} \right)^2 - 1
\]

When \(b \to \infty\), \(\varphi(t) = \frac{e^2}{2}\). Also
\[
\psi(t) = \left| t^\epsilon \right| \left( \int_a^b u^{-\epsilon} e^u du \right) = e^2 \left( \frac{t}{b} \right)^2 - 1 = \frac{e^2}{2}
\]

and
\[
\vartheta(t) = \left| t^\epsilon \right| \left( \int_a^b u^{-\epsilon} e^u du \right) = e^2 \frac{2}{e^2}
\]

as \(b \to \infty\). Hence,
\[
\left\| f(t) - f_0(t) \right\| \leq \left\| t^\epsilon \right\| \left( \int_a^b u^{-\epsilon} e^u du \right) = \frac{e^2}{4} \left( \frac{t}{b} \right)^2 - 1
\]

When \(b \to \infty\),
\[
\left\| f(t) - f_0(t) \right\| \leq \frac{e^2}{4},
\]

where
\[
f_0(t) = \left[ \left( \frac{2x_0 - x - x_2}{a} \right) + \left( \frac{x_0 - x_2}{a} \ln \left(\frac{t}{a}\right) \right) t \right] + \left[ \left( \frac{x + x_2 - 2x_0}{a^2} \right) + \left( \frac{x_0}{a^2} \ln \left(\frac{t}{a}\right) \right) t^2,\right.
\]

and \(x, \xi, x_0, x_1, x_2, x_3, x_4\) and \(x_5\) are the unique elements of \(X\).

**Acknowledgment**

The authors are thankful to the referee for suggesting the publication of this article.

**References**

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